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THEORY OF NONLINEAR HEAT AND MASS TRANSFER ON A POROUS SEMIINFINITE PLATE

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The nonlinear transport process (thermal conductivity or diffusion) is considered in a viscous liquid flowing near the plane of a semiinfinite plate. It is shown that under certain conditions there is rigorous spatial localization of the thermal or diffusive boundary layer.

Let the stationary flow of a Newtonian viscous liquid move over the plane of a semiinfinite plane $x \geqslant y$, $y = 0$, (Fig. 1) in the positive direction of the x axis. We assume that the velocity distribution at the external boundary of the laminar boundary layer formed over the plate is described by the expression $U = cx^m$, where c and m are constants$\geqslant 0$ (one-parameter class of boundary-layer theory [1]). For the sake of generality, it is also assumed that on the surface of the plate there is inhomogeneous fluid blowing or suction, proportional to $x^{(m-1)/2}$. It is assumed that on the surface of the plate there is heat transfer or isothermal diffusion of the plate material in the leading flow, and the corresponding transport coefficient χ depends on the transfer characteristic f(x, y) (temperature or concentration) according to the power law

$$\chi = an\left(\frac{f}{f_w}\right)^{n-1};\ a,\ n,\ f_w - \text{const} > 0.$$

Here and below the subscript w denotes the value of the corresponding quantity at the surface of the plate.

In the boundary-layer theory approximation the nonlinear transport process under consideration is described by the system of equations [2]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2};\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \frac{a}{f_w^{n-1}}\frac{\partial^2 f}{\partial y^2}. \tag{2}$$

Here u(x, y) and v(x, y) are the longitudinal and transverse components of the fluid velocity.

Assuming that there is no transferable characteristic in the leading flow ("vanishing background"), the boundary conditions which the solution of system (1), (2) must satisfy are written in the form

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$$u(x, 0) = 0,\ v(x, 0) = -\frac{m+1}{2}(c\nu x^{m-1})^{1/2}\varphi_w,\ u(x, \infty) = cx_m, \tag{3}$$

$$f(x, 0) = f_w,\ f(x, \infty) = \nabla f^n(x, \infty) = 0. \tag{4}$$

Here φ_w = const is the blowing parameter; $\varphi_w < 0$ in the case of blowing and $\varphi_w > 0$ in the case of suction. (The last condition in (4) expresses the absence of a flow of transfer characteristics upon being far removed from the wall).

It follows from physical considerations that everywhere at x, $y \geqslant 0$ the functions f(x, y) and $\nabla f^n(x, y)$ must be continuous.

The functions u(x, y) and v(x, y) are determined as a solution of the self-similar problem (1), (3) [2]:

$$u(x, y) = cx^m\varphi'(\eta),$$

$$v(x, y) = \frac{1}{2}(c\nu x^{m-1})^{1/2}[(m-1)\eta\varphi'(\eta) - (m+1)\varphi(\eta)].$$

Here and below the prime denotes ordinary differentiation with respect to the self-similar variable $\eta = y(cx^{m-1}/\nu)^{1/2}$.

Tables of values of the functions $\varphi(\eta)$, being the solution of the boundary-value problem for the ordinary differential equation, are given in [3]. It is important to note that $\varphi(\eta)$ is an increasing function of η, while $\varphi(0) = \varphi_w$, $\varphi'(\infty) = 1$.

A solution f(x, y) of problem (2), (4) is sought in the form

$$f(x, y) = f_w\Theta(\eta). \tag{5}$$

Substituting (5) into (2) and (4), we have

$$[\Theta^n(\eta)]'' + \frac{1}{2}\sigma\varphi(\eta)\Theta'(\eta) = 0;\ \Theta(0) = 1,\ \Theta(\infty) = (\Theta^n)'(\infty) = 0. \tag{6}$$

Here $\sigma = (m + 1)\nu/2a$ is the Prandtl number (thermal or diffusive depending on the transport process under consideration).

The solution of problem (6) cannot be obtained analytically, and numerical methods must be used to solve it. It is recommended, however, to start with a preliminary qualitative study of the properties of the solution of problem (6), which we provide initially for the case $\varphi(0) \equiv \varphi_w \geqslant 0$, when $\varphi(\eta) > 0$ for all $0 < \eta < \infty$ [2, 3].

We introduce the new unknown function $\psi(\eta) \equiv \Theta^n(\eta)$, which is defined by the following problem:

$$\psi''(\eta) = -\omega(\eta)[\psi^{1/n}(\eta)]',\ \omega(\eta) = \frac{1}{2}\sigma\varphi(\eta), \tag{7}$$

$$\psi(0) = 1,\ \psi(\infty) = \psi'(\infty) = 0. \tag{8}$$

We note that only nonnegative solutions of problem (6), i.e., $\Theta(\eta) \geqslant 0$, $\psi(\eta) \geqslant 0$, have a physical meaning. It therefore follows from (7) that $\psi'(\eta)$ and $\psi''(\eta)$ have different signs for $0 < \eta < \infty$. Taking into account the condition $\psi(\infty) = 0$ and the fact that $\psi'(\eta)$ is a continuous function, it can be concluded that $\psi(\eta)$ is a convex, monotonic, decreasing function.

Let $\eta = \eta_* > 0$ be some fixed value of η, so that $\psi(\eta_*) \equiv \psi_* > 0$. Obviously, the solution of problem (7), (8) for $\eta > \eta_*$ can be considered as a solution of Eq. (7) with conditions

$$\psi(\eta_*) = \psi_*,\ \psi(\infty) = \psi'(\infty) = 0. \tag{9}$$

Consider the auxiliary problem

$$\bar\psi''(\eta) = -\omega_*[\bar\psi^{1/n}(\eta)]',\ \bar\psi(\eta_*) = \psi_*,\ \bar\psi(\infty) = \bar\psi'(\infty) = 0, \tag{10}$$

differing from (7), (9) in that $\omega(\eta)$ in Eq. (7) is replaced by the constant $\omega_* \equiv \omega(\eta_*)$ = const. The auxiliary problem (10) has an analytic solution, whose form varies with n. For $n < 1$ it is written in the form

$$\bar\psi(\eta) = \psi_*\left[1 + \frac{1-n}{n}\omega_*\psi_*^{\frac{1-n}{n}}(\eta - \eta_*)\right]^{\frac{n}{n-1}},\ \eta_* < \eta < \infty,$$

while for n = 1,

$$\bar{\psi}(\eta) = \psi_* e^{-\omega_*(\eta-\eta_*)}, \quad \eta_* < \eta < \infty.$$

For n > 1 the solution of problem (10) must not be represented in a single analytic form for all values $0 < \eta < \infty$. For n > 1, however, Eq. (10) has the singular solution $\bar{\psi}(\eta) \equiv 0$ [4], and this makes it possible to construct the generalized solution of problem (10) in the form

$$\bar{\psi}(\eta) = \begin{cases} \psi_* \left(\dfrac{\bar{\eta}_f - \eta}{\bar{\eta}_f - \eta_*} \right)^{\frac{n}{n-1}} & \text{for } \eta_* < \eta < \bar{\eta}_f, \\ 0 & \text{for } \bar{\eta}_f < \eta < \infty, \end{cases} \tag{11}$$

$$\bar{\eta}_f = \eta_* + \frac{n}{n-1} \frac{\psi^{\frac{n-1}{n}}}{\omega_*}.$$

It is easily verified that the constructed solution satisfies all conditions of problem (10), as well as the physical requirement of continuity of the functions f, ∇f^n for all $\eta \geq \eta_*$.

We now show that for $\eta > \eta_*$ the solution of (10) $\bar{\psi}(\eta)$ is an upper bound of the solutions of problem (7), (9), $\psi(\eta)$, i.e., the following inequality holds

$$\psi(\eta) \leqslant \bar{\psi}(\eta), \quad \eta_* < \eta < \infty. \tag{12}$$

Putting $\alpha(\eta) \equiv \psi(\eta) - \bar{\psi}(\eta)$, we have from (7), (9), and (10)

$$\alpha''(\eta) + \omega_* [\psi^{1/n}(\eta) - \bar{\psi}^{1/n}(\eta)]' = -[\omega(\eta) - \omega_*][\psi^{1/n}(\eta)]', \tag{13}$$

$$\alpha(\eta_*) = \alpha(\infty) = \alpha'(\infty) = 0. \tag{14}$$

Transforming $\psi^{1/n}(\eta) - \bar{\psi}^{1/n}(\eta)$ by the Lagrange equation, and integrating Eq. (13) once with account of the two last conditions (14), we obtain

$$\alpha'(\eta) + P(\eta)\alpha(\eta) = Q(\eta),$$

$$P(\eta) \equiv \frac{1}{n} \omega_* [\beta\bar{\psi} - (1-\beta)\psi]^{\frac{1-n}{n}}, \tag{15}$$

$$Q(\eta) \equiv \int_\eta^\infty [\omega(\eta) - \omega_*] \; [\psi^{1/n}(\eta)]' \, d\eta.$$

The solution (15) **satisfying the first condition** of (14) **is written in the form**

$$\alpha(\eta) = \exp\left[-\int_{\eta_*}^{\infty} P(\xi)\, d\xi\right] \int_{\eta_*}^{\eta} Q(\zeta) \exp\left[\int_{\eta_*}^{\xi} P(\varkappa)\, d\varkappa\right] d\zeta,$$

while the function $\alpha(\eta)$ is bounded, since the functions $\psi(\eta)$ and $\bar{\psi}(\eta)$ are bounded. Since $\omega(\eta) - \omega_* > 0$, and $\psi'(\eta) < 0$, it follows that $Q(\eta) < 0$ and, as a consequence, $\alpha(\eta) < 0$.

Thus, inequality (12) is proved.

Taking into account the structure of solution (11) and of inequality (12), it can be concluded that for n > 1 the function $\psi(\eta)$ vanishes together with $\psi'(\eta)$ at some value $\eta = \eta_f < \bar{\eta}_f$. The solution of problem (7), (9) and, consequently, the solution of problem (7), (8) differs from zero at $\eta < \eta_f$ and vanishes at $\eta = \eta_f$, while at $\eta > \eta_f$ the singular solution of Eq. (7) is continued to be zero. In other words, for n > 1 there exists a surface $y = y_f(x) \equiv \eta_f(\nu x^{1-m}/c)^{1/2}$, η_f = const, separating the regions in which f(x, y) = 0 and f(x, y) ≠ 0, i.e., there is rigorous spatial localization of the thermal or diffusion boundary layer.

We also note that for n > 1 one can obtain a description of the behavior of the solution of problem (6) near the indicated surface $\eta = \eta_f$ = const. Replacing in (6) the function $\varphi(\eta)$ by the constant $\varphi(\eta_f)$, we arrive at the equation

$$[\Theta^n(\eta)]'' + \frac{1}{2}\sigma\varphi(\eta_f)\Theta'(\eta) = 0, \tag{16}$$

whose solution must satisfy the following conditions on the surface $\eta = \eta_f$:

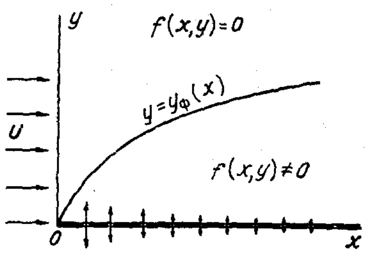

Fig. 1. Stationary boundary layer over a semiinfinite plate.

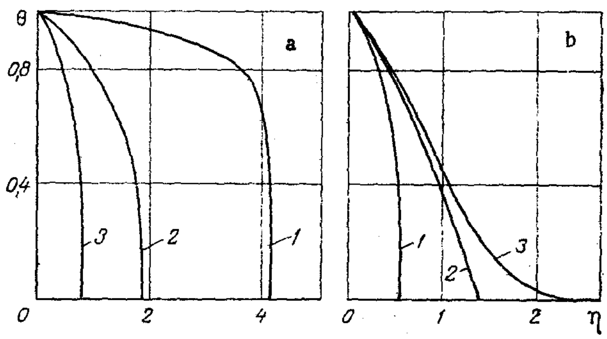

Fig. 2. Distribution of transport characteristics for various values: a) blowing parameter φ_w ($\sigma = 1$; $n = 2.5$; 1) $\varphi_w = -1$; 2) 0; 3) 1); b) parameter n ($\varphi_w = 0$; $\sigma = 1$; 1) $n = 2.5$; 2) 2; 3) 1.5).

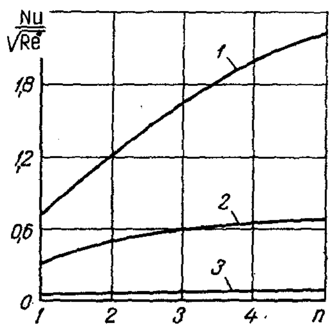

Fig. 3. The local Nusselt number as a function of the parameter n ($\sigma = 1$): 1) $\varphi_w = -1$; 2) 0; 3) 1.

$$\Theta(\eta_f) = [\Theta^n(\eta_f)]' = 0. \tag{17}$$

Integrating Eq. (16) twice with account of condition (17), we obtain the expression

$$\Theta(\eta) = \left[\frac{n-1}{2n}\,\sigma\varphi(\eta_f)(\eta_f - \eta)\right]^{\frac{1}{n-1}}, \tag{18}$$

determining the transport characteristic near the surface $\eta = \eta_f = \text{const}$ for $\eta < \eta_f$. For $\eta > \eta_f$ its distribution is determined by the singular solution of (16) $\Theta(\eta) = 0$, which exists if $n > 1$ [4].

All the arguments above for the case $\varphi_w > 0$ can also be repeated for the case $\varphi_w < 0$ (blowing), since it follows from the properties of the function $\varphi(\eta)$ that there exists a point $\eta = \eta_0 < \infty$, such that $\varphi(\eta) > 0$ for all $\eta > \eta_0$.

Problem (6) was solved numerically by an iteration method [5], where the iteration process consisted of the following scheme:

$$n\left[\left(\frac{\Theta_j^i+\Theta_{j+1}^i}{2}\right)^{n-1}\frac{\Theta_{j+1}^{i+1}-\Theta_j^{i+1}}{h}-\left(\frac{\Theta_{j-1}^i+\Theta_j^i}{2}\right)^{n-1}\frac{\Theta_j^{i+1}-\Theta_{j-1}^{i+1}}{h}\right]\frac{1}{h}+\frac{1}{2}\sigma\varphi_j\,\frac{\Theta_{j+1}^{i+1}-\Theta_{j-1}^{i+1}}{2h}=0.$$

As to $\Theta_{(j+1)}^{i+1}$, $\Theta_{(j)}^{i+1}$, $\Theta_{(j-1)}^{i+1}$, the difference scheme is linear, and therefore the value of $\Theta_{(\,)}^{i+1}$ can be found from $\Theta_{(\,)}^{i}$, well known by the sweep method. As the initial iteration we chose the $\Theta_{(\,)}^{0}$ value for n = 1 [3].

The distribution of transferable characteristics $\Theta(\eta)$ is shown in Fig. 2a, b. The curves provided illustrate the described effect of spatial localization of the thermal or diffusion boundary layer for $n > 1$.

Figure 2a shows the effect of the blowing (suction) parameter φ_w on the distribution of transport characteristics. The curves of Fig. 2b illustrate the different behavior of the function $\Theta(\eta)$ near the separating surfaces as a function of $n > 1$. Thus, e.g., $\Theta'(\eta_f) = 0$ for n = 1.5, $\Theta'(\eta_f) < 0$ and is bounded for n = 2, and $\Theta'(\eta_f) = -\infty$ for n = 3, which agrees with expression (18) for the solution of problem (6).

In boundary-layer theory the intensity of the heat-or mass-transfer process from the surface of the plate to the fluid at a distance x from the front edge of the plate is usually characterized by the ratio of the local thermal or diffusion Nusselt number Nu_x to the local Reynolds number Re_x [6]. Figure 3 shows the dependence of the ratio Nu/Re_x on the value of n in the expressions for the transport coefficient, calculated for different values of the injection parameter φ_w.

In conclusion, we point out that rigorous spatial localization of a stationary thermal or diffusion boundary layer can also be observed in considering nonpolar two-dimensional and three-dimensional problems if the corresponding transport coefficients depend on the transport characteristics.

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